

# Math 246A Lecture 1 Notes

Daniel Raban

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## 1 Complex Numbers, Elementary Mappings, and the Fundamental Theorem of Algebra

### 1.1 The complex numbers

Consider the set  $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ . We can express this set differently, as the complex numbers.

**Definition 1.1.** The set of **complex numbers** is  $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ .

**Definition 1.2.** **Addition** of complex numbers is defined component-wise (usual vector addition):

$$(x + iy) + (u + iv) := (x + u) + i(y + v).$$

**Multiplication** of complex numbers is defined as

$$(x + iy)(u + iv) := (xu - yv) + i(xv + yu).$$

With these operations,  $\mathbb{C}$  is a **field**.

**Definition 1.3.** Let  $z = x + iy$ . The **complex conjugate** of  $z$  is  $\bar{z} = x - iy$ .

The map taking  $z \mapsto \bar{z}$  is reflection about the real axis in the complex plane. Note that  $z$  is real ( $y = 0$ ) if and only if  $z = \bar{z}$ .

**Definition 1.4.** The **absolute value** (or **modulus**) of  $z$  is  $|z| = \sqrt{x^2 + y^2}$ .

Note that  $z\bar{z} = |z|^2$ . So if  $z \neq 0$ , then  $\bar{z}/|z|^2 = z^{-1}$ . If  $z \neq 0$ , then  $\left|\frac{z}{|z|}\right| = 1$ . So there exists some  $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$  such that  $z/|z| = \cos(\alpha) + i\sin(\alpha)$ .

**Definition 1.5.** Let  $z = |z|(\cos(\alpha) + i\sin(\alpha))$ . Then  $\alpha$  is called the **argument** of  $z$  (sometimes denoted  $\arg(z)$ ).

**Proposition 1.1.** Let  $z, w \in \mathbb{C}$ . Then

$$|zw| = |z| \cdot |w|, \quad \arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}.$$

*Proof.* The first assertion follows from an algebraic calculation. For the second assertion, recall the trigonometric identities

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

Combining these, we get that

$$(\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) = \cos(\alpha + \beta) + i\sin(\alpha + \beta). \quad \square$$

So the geometric picture of multiplication of complex numbers is that the lengths  $|z|$  and  $|w|$  multiply, and the angles  $\arg(z)$  and  $\arg(w)$  add.

## 1.2 Elementary mappings

Let's look at what certain maps  $f : \mathbb{C} \rightarrow \mathbb{C}$  look like.

1.  $z \mapsto \bar{z}$ : This is a reflection over the real axis in the complex plane. Note that  $\overline{(\bar{z})}$ , so this map is an involution.
2.  $z \mapsto z^2$ : This takes a circle of radius  $R$  about 0 to a circle of radius  $R^2$  about zero. If  $R > 1$ , then the radius grows; if  $R < 1$ , then the radius shrinks. Additionally, if  $\arg(z) = \alpha$ , then  $\arg(z^2) = 2\alpha$ , so this map takes a circle and wraps it around the image circle twice.
3.  $z \mapsto z^m$  for  $m = 2, 3, 4, \dots$  : Since  $(\cos(\alpha) + i\sin(\alpha))^m = \cos(m\alpha) + i\sin(m\alpha)$ , this wraps a circle around the corresponding image circle  $m$  times.
4.  $z \mapsto 1/z$ :  $|\frac{1}{z}| = \frac{1}{|z|}$ , so this map sends a circle of radius  $R$  to a circle of radius  $1/R$ . This map sends  $\cos(\alpha) + i\sin(\alpha) \mapsto \cos(\alpha) - i\sin(\alpha)$  (think  $1/i = -i$ ), so a circle being traversed counterclockwise gets mapped to a circle being traversed clockwise.
5.  $z \mapsto 1/\bar{z}$ :  $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$ , so  $\overline{\left(\frac{1}{z}\right)} = \frac{x+iy}{x^2+y^2} = \frac{z}{|z|^2}$ . So this map inverts the modulus of  $z$  but doesn't change the angle.

## 1.3 The fundamental theorem of algebra

**Theorem 1.1.** *If  $p(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$  with  $a_i \in \mathbb{C}$ ,  $a_n \neq 0$ ,  $n \geq 1$ , then there exists some  $z_0 \in \mathbb{C}$  such that  $p(z_0) = 0$ .*

Before we prove this, we need a lemma.

**Lemma 1.1.** *There exists  $R_0 > 0$  such that if  $|z| > R_0$ , then  $|p(z)| > \frac{|a_n|}{2}|z|^n > a_0$ .*

*Proof.* First, take  $R_0 > \left(\frac{2|a_0|}{|a_n|}\right)^{1/n}$  and assume  $R_0 > 1$ . Now, also take  $R_0$  to satisfy

$$\frac{|a_0| + \cdots + |a_{n-1}|}{|a_n|} < \frac{R_0}{2}.$$

Now, for  $|z| > R_0$ ,  $|z^j| > R_0^j > 1$ . So

$$\begin{aligned} |p(z)| &\geq |a_n||z^n| - (|a_0| + \cdots + |a_{n-1}|)|z|^{n-1} \\ &\geq |a_n||z^n| - \frac{R_0}{2}|a_n||z|^{n-1} \\ &\geq |a_n||z|^n - \frac{|a_n|}{2}|z|^n \\ &= \frac{|a_n|}{2}|z|^n. \end{aligned} \quad \square$$

Now, we prove the fundamental theorem of algebra. This proof, unlike proofs we will be able to give later, is very elementary.

*Proof.* We can assume that  $a_0 \neq 0$  and  $n \geq 2$ . Let  $R_0$  be as in the lemma, and let  $K = \{z \in \mathbb{C} : |z| \leq R_0\}$ . Outside of  $K$ ,  $|z| > R_0$ , so  $|p(z)| > |a_0| > 0$ . So all zeros of  $p$  must be inside  $K$ . Since  $K$  is compact and  $p : K \rightarrow \mathbb{C}$  is continuous, there exists some  $z_0 \in K$  such that  $|p(z_0)| = \inf\{|p(z)| : z \in K\}$ .

For contradiction, assume there is no zero of  $p$  in  $K$ , and let  $C_\delta = \{z : |z - z_0| = \delta\} \subseteq K$  be the circle of radius  $\delta$  around  $z_0$  for some small  $\delta > 0$ ; we will choose  $\delta$  later. Let  $b_0 = p(z_0) \neq 0$ . Then  $p(z) = b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^k$ , where  $b_n = a_n$ . Take  $1 \leq k \leq n$  least such that  $b_k \neq 0$ , so

$$p(z) = b_0 + \underbrace{b_k(z - z_0)^k}_{Q(z)} + \underbrace{\sum_{j=k+1}^n b_j(z - z_0)^j}_{R(z)}$$

Now observe that  $z \mapsto b_0 + Q(z)$  maps  $C_\delta$  to a circle, and wraps around it  $k$  times. If  $\delta < 1$ , then

$$|R(z)| \leq \sum_{j=k+1}^n |b_j|\delta^{k+1} \leq \sum_{j=k+1}^n |b_j|\frac{\delta}{|b_k|}|Q(z)|.$$

Now pick  $\delta$  small enough that  $(\sum_{j=k+1}^n |b_j|)\delta \leq |b_k|/2$ . Then  $|R(z)| \leq |Q(z)|/2$ . Since  $z \mapsto b_0 + Q(z)$  wraps around the circle  $b_0 + Q(C_\delta)$   $k \geq 1$  times, we can pick the  $z$  that gets mapped closest to the origin (making  $\delta$  small enough that this image circle does not contain the origin in its interior). This will give us a contradiction because we will get a point in the image of  $p$  that has smaller magnitude than  $z_0$ , contradicting the minimality of  $z_0$ . We will make this precise next lecture.  $\square$